Pergamon

J. Appl. Maths Mechs, Vol. 58, No. 4, pp. 593–600, 1994

0021–8928/94 $24.00+0.00

0021-8928(94)00086-7

EXISTENCE OF THE VALUE OF A MANY-PERSON GAME OF PURSUIT†

N. N. PETROV

Izhevsk

(*Received* 21 *October* 1993)

It is proved that a game of pursuit has a value in a class of piecewise open-loop strategies (POLS) defined somewhat differently from those considered previously for two players [1], in differential games with several evaders and pursuers.

Various aspects of the existence of the value of a two-person game were considered in [2–6], where an extensive bibliography can be found. The problem of approach–departure with several target sets was considered in [7], where the problem of the existence of an equilibrium position was solved not in the class of POLS, as is done below, but in the class of piecewise positional strategies.

1. STATEMENT OF THE PROBLEM

We consider a problem of pursuit with bounded time T between a team of pursuers $P=\{P_1, \ldots, P_n\}$ and a team of evaders $E=\{E_1, \ldots, E_m\}$, which we shall treat as a zero-sum two-person game between P and E.

Suppose that the equations of motion of players P_i (the pursuers, $i=1, 2, \ldots, n$) and E_j (the evaders, $j=1, 2, \ldots, m$) are

$$\begin{gathered} P_i: \dot{\mathbf{x}}_i = \mathbf{f}_i(\mathbf{x}_i, \mathbf{u}_i),\ \mathbf{u}_i \in U_i,\ \mathbf{x}_i(0) = \mathbf{x}_i^0 \\ E_j: \dot{\mathbf{y}}_j = \mathbf{g}_j(\mathbf{y}_j, \mathbf{v}_j),\ \mathbf{v}_j \in V_j;\ \mathbf{y}_j(0) = \mathbf{y}_j^0 \\ \mathbf{x}_i, \mathbf{y}_j \in R^k,\ U_i \subset R^{n_i},\ V_j \subset R^{m_j} \end{gathered} \tag{1.1}$$

where U_i and V_j are compact sets.

Let $\mathbf{X}_0=(\mathbf{x}_1^0, \ldots, \mathbf{x}_n^0)$, $\mathbf{Y}_0=(\mathbf{y}_1^0, \ldots, \mathbf{y}_m^0)$. We let $\Gamma(\mathbf{X}_0, \mathbf{Y}_0, J_T)$ denote the game which begins at time $t=0$ from initial positions $(\mathbf{X}_0, \mathbf{Y}_0)$ with payoff function J_T. In the class of POLS, in the case $m=n=1$, it has been proved [1] that an ε-equilibrium situation exists in this game when the payoff function is

$$J_T = \min_{t\in[0,T]} \|\mathbf{x}(t) - \mathbf{y}(t)\|$$

Below, using a similar method and a slightly modified definition of POLS, we shall prove the analogous assertion for the function

†*Prikl. Mat. Mekh.* Vol. 58, No. 4, pp. 22–29, 1994.

$$J_T = \sum_j \min_{t\in[0,T]} \min_i \|\mathbf{x}_i(t) - \mathbf{y}(t)\|$$

Here, as later, summation over j runs from 1 to m, and over j, from 1 to n.

Some corollaries of this result are relevant for games of pursuit with unbounded time.

We shall consider system (1.1) under the following assumptions:

(a) the functions $\mathbf{f}_i(\mathbf{x}_i, \mathbf{u}_i)(\mathbf{g}_i(\mathbf{y}_i, \boldsymbol{\upsilon}_j))$ are defined and continuous $(\mathbf{x}_i, \mathbf{u}_i) \in R^k \times U_i(y_j, \boldsymbol{\upsilon}_j) \in R^k \times V_j)$;

(b) the functions $\mathbf{f}_i(\mathbf{x}_i, \mathbf{u}_i)(\mathbf{g}_i(\mathbf{y}_i, \boldsymbol{\upsilon}_j))$ satisfy a local Lipschitz condition as functions of $\mathbf{x}_i(\mathbf{y}_j)$ with a constant independent of $\mathbf{u}_i(\boldsymbol{\upsilon}_j)$;

(c) for all $(\mathbf{x}_i, \mathbf{u}_i)(\mathbf{y}_i, \boldsymbol{\upsilon}_j)$

$$|\langle \mathbf{x}_i, \mathbf{f}_i(\mathbf{x}_i, \mathbf{u}_i)\rangle| \leq C_i\left(1 + \|\mathbf{x}_i\|^2\right),\ |\langle \mathbf{y}_j, \mathbf{g}_j(\mathbf{y}_j, \boldsymbol{\upsilon}_j)\rangle| \leq D_j\left(1 + \|\mathbf{y}_j\|^2\right)$$

Then for any choice of measurable functions $\mathbf{u}_i = \mathbf{u}_i(t)$, $\boldsymbol{\upsilon}_j = \boldsymbol{\upsilon}_j(t)$ defined in $[0, T]$ with values in U_i and V_j, systems (1.1) have solutions $\mathbf{x}_i = \mathbf{x}_i(t, \mathbf{x}_i^0, \mathbf{u}_i(t))$, $\mathbf{y}_j = \mathbf{y}_j(t, \mathbf{y}_j^0, \boldsymbol{\upsilon}_j(t))$ with initial data $(0, \mathbf{x}_i^0)$, $(0, \mathbf{y}_j^0)$, defined in $[0, T]$. Moreover, numbers R and δ_0 exist such that the inequalities

$$\|\mathbf{x}_i(t, \mathbf{x}_i, \mathbf{u}_i(t))\| \leq R,\ \|\mathbf{y}_j(t, \mathbf{y}_j, \boldsymbol{\upsilon}_j(t))\| \leq R$$

hold for any measurable functions $\mathbf{u}_i = \mathbf{u}_i(t) \in U_i$, $\boldsymbol{\upsilon}_j = \boldsymbol{\upsilon}_j(t) \in V_j$ defined in $[0, T]$ and any $\mathbf{x}_i \in D_{\delta_0}(\mathbf{x}_i^0)$, $\mathbf{y}_j \in D_{\delta_0}(\mathbf{y}_j^0)$. In addition, constants $L = L(D_R(0))$ and M exist such that, for any $\mathbf{x}_i^1$, $\mathbf{x}_i^2 \in D_{\delta_0}(\mathbf{x}_i^0)$, $\mathbf{y}_j^1$, $\mathbf{y}_j^2 \in D_{\delta_0}(\mathbf{y}_j^0)$, t_1, $t_2 \in [0, T]$, and any measurable functions $\mathbf{u}_i = \mathbf{u}_i(t) \in U_i$, $\boldsymbol{\upsilon}_j = \boldsymbol{\upsilon}_j(t) \in V_j$

$$\|\mathbf{x}_i(t, \mathbf{x}_i^1, \mathbf{u}_i(t)) - \mathbf{x}_i(t, \mathbf{x}_i^2, \mathbf{u}_i(t))\| \leq e^{LT}\|\mathbf{x}_i^1 - \mathbf{x}_i^2\|$$

$$\|\mathbf{x}_i(t_1, \mathbf{x}_i, \mathbf{u}_i(t)) - \mathbf{x}_i(t_2, \mathbf{x}_i, \mathbf{u}_i(t))\| \leq M|t_1 - t_2| \tag{1.2}$$

and analogous inequalities hold with $\mathbf{x}_i$ and $\mathbf{u}_i$ replaced by $\mathbf{y}_j$ and $\boldsymbol{\upsilon}_j$.

We will now describe the game-theoretic elements of the problem. By a finite partition σ of $[0, T]$ we mean a collection of distinct numbers $\{0, T, t_l \in (0, T), l = 1, \ldots, r\}$, indexed in increasing order. The set of all σ will be denoted by Σ. Every partition $\sigma \in \Sigma$ generates a partition $\sigma_l = \{t_l, \ldots, T\}$ of the interval $[t_1, T]$.

Definition 1. A piecewise open-loop strategy (POLS) Q_i for player P_i is a pair (σ, Q_σ), $\sigma \in \Sigma$

$$\Sigma :\ 0 = t_0 < t_1 < t_2 < \ldots < t_r < t_{r+1} = T \tag{1.3}$$

where Q_σ is the set of mappings $b_i^l (l = 0, 1, \ldots, r)$ that produce, given the quantities

$$(t_l, \mathbf{x}_1(t_l), \ldots, \mathbf{x}_n(t_l), \mathbf{y}_1(t_l), \ldots, \mathbf{y}_m(t_l))$$

$$\min_{t\in[0,t_l]} \min_i \|\mathbf{x}_i(t) - \mathbf{y}_1(t)\|, \ldots, \min_{t\in[0,t_l]} \min_i \|\mathbf{x}_i(t) - \mathbf{y}_m(t)\|) \tag{1.4}$$

a measurable function $\mathbf{u}_i = \mathbf{u}_i(t) \in U$ defined for $t \in [t_l, t_{l+1})$. A similar definition yields a POLS S_j for player E_j. The sequence $(Q_1, \ldots, Q_n, S_1, \ldots, S_m)$ will be called a situation. Under our assumptions, in every situation $(Q_1, \ldots, S_m)$ we can define trajectories of motion $\mathbf{x}_i(t)\mathbf{y}_j(t)$ for $t \in [0, T]$, so that we can define the value of the payoff function

$$K(Q_1, \ldots, Q_n, S_1, \ldots, S_m) = \sum_j \min_{t\in[0,T]} \min_i \|\mathbf{x}_i(t) - \mathbf{y}_j(t)\| \tag{1.5}$$

Player P aims to minimize the quantity $K(Q_1, \ldots, S_m)$ while player E aims to maximize it.

Consider the following sets

$$G_{P_i}^{t_l} = \bigcup_{\mathbf{x}_i \in D_{\delta_{t_l}}(\mathbf{x}_i^0)} \bigcup_{0 \le t \le t_l} C^t(\mathbf{x}_i),\quad G_{E_j}^{t_l} = \bigcup_{\mathbf{y}_j \in D_{\delta_0}(\mathbf{y}_j^0)} \bigcup_{0 \le t \le t_l} C^t(\mathbf{y}_j)$$

where $C^t(\mathbf{x}_i)$ and $C^t(\mathbf{y}_j)$ are the sets of points that players P_i and E_j may reach at time t, having begun to move at the initial time from $\mathbf{x}_i$, $\mathbf{y}_j$ along trajectories of system (1.1).

Let $A_i(\mathbf{x}_i, \tau)$ and $A_j(\mathbf{y}_j, \tau)$ be the set of all trajectories of players P_i and E_j defined in $[0, \tau]$ and starting at $\mathbf{x}_i$, $\mathbf{y}_j$, and let $\overline{A_i(\mathbf{x}_i \tau)}$ and $\overline{A_j(\mathbf{y}_j \tau)}$ be the closures of these sets in the space of continuous functions.

We shall need the following notation

$$W_0 = D_{\delta_0}(\mathbf{x}_1^0) \times \ldots \times D_{\delta_0}(\mathbf{x}_n^0) \times D_{\delta_0}(\mathbf{y}_1^0) \times \ldots \times D_{\delta_0}(\mathbf{y}_m^0)$$

$$W_l = G_{P_1}^{t_l} \times \ldots \times G_{P_n}^{t_l} \times G_{E_1}^{t_l} \times \ldots \times G_{E_m}^{t_l}$$

$$\mathbf{X} = (\mathbf{x}_1, \ldots, \mathbf{x}_n),\ \mathbf{Y} = (\mathbf{y}_1, \ldots, \mathbf{y}_m),\ \mathbf{x}_i, \mathbf{y}_j \in R^k$$

$$\mathbf{R} = (\rho_1, \ldots, \rho_m),\ \rho_j \in R^1,\ \rho_j^0 = \min_i \|\mathbf{x}_i^0 - \mathbf{y}_j^0\|$$

$$\|\mathbf{X}_1 - \mathbf{X}_2\| = \sum_i \|\mathbf{x}_i^1 - \mathbf{x}_i^2\|,\ \|\mathbf{Y}_1 - \mathbf{Y}_2\| = \sum_j \|\mathbf{y}_j^1 - \mathbf{y}_j^2\|,\ \|\mathbf{R}_1 - \mathbf{R}_2\| = \sum_j |\rho_j^1 - \rho_j^2|$$

$$\overline{A(\mathbf{X}, \tau)} = \overline{A_1(\mathbf{x}_1, \tau)} \times \overline{A_2(\mathbf{x}_2, \tau)} \times \ldots \times \overline{A_n(\mathbf{x}_n, \tau)}$$

$$\overline{A(\mathbf{Y}, \tau)} = \overline{A_1(\mathbf{y}_1, \tau)} \times \ldots \times \overline{A(\mathbf{y}_n, \tau)}$$

The game starting at initial position $(\mathbf{X}_0, \mathbf{Y}_0)$, with payoff function (1.4), in which players P and E may use POLS, will be denoted by $\Gamma(\mathbf{X}_0, \mathbf{Y}_0)$, and its value by $V(\mathbf{X}_0, \mathbf{Y}_0)$.

2. SOME AUXILIARY GAMES

We will consider a game $\Gamma(\mathbf{X}_0, \mathbf{Y}_0, \sigma_0)$ which differs from $\Gamma(\mathbf{X}_0, \mathbf{Y}_0)$ only in the information state of the players and the class of admissible strategies. Let $\sigma_0 \in \Sigma$ (1.3). In the game $\Gamma(\mathbf{X}_0, \mathbf{Y}_0, \sigma_0)$, players E_j use POLS as in Definition 1.

Definition 2. A piecewise open-loop co-strategy Q_i of player P_i in the game $\Gamma(\mathbf{X}_0, \mathbf{Y}_0, \sigma_0)$ is a family of mappings $b_i^l (l = 0, 1, \ldots, r)$ that produce, given the quantities (1.4) and controls $\mathbf{v}_j(t)$, $t \in [t_l, t_{l+1})$, a measurable function $\mathbf{u}_i = \mathbf{u}_i(t) \in U_i$ defined for $t \in [t_l, t_{l+1})$.

Define numbers $\underline{V}(\mathbf{X}_l, \mathbf{Y}_l, \sigma_l, \mathbf{R}_l)$ as follows:

$$\begin{gathered}
\underline{V}(\mathbf{X}_l, \mathbf{Y}_l, \mathbf{R}_l, \sigma_l) = \max_{\mathbf{Y}_{l+1}(t) \in \overline{A(\mathbf{Y}_l, \Delta t_l)}} \min_{\mathbf{X}_{l+1}(t) \in \overline{A(\mathbf{X}_l, \Delta t_l)}} \underline{V}(\mathbf{X}_{l+1}(\Delta t_l), \mathbf{Y}_{l+1}(\Delta t_l), \mathbf{R}_{l+1}, \sigma_{l+1}) \\
(\mathbf{X}_l, \mathbf{Y}_l) \in W_l,\ \Delta t_l = t_{l+1} - t_l,\ \mathbf{R}_{l+1} = (\rho_1^{l+1}, \ldots, \rho_m^{l+1}) \\
\rho_j^{l+1} = \min\left\{\rho_j^l, \min_{t \in [0, \Delta t_l]} \min_i \|\mathbf{x}_i^{l+1}(t) - \mathbf{y}_j^{l+1}(t)\|\right\},\ l = 0, \ldots, r-1 \\
\underline{V}(\mathbf{X}_r, \mathbf{Y}_r, \mathbf{R}_r, \sigma_r) = \max_{\mathbf{Y}_{r+1}(t) \in \overline{A(\mathbf{Y}_r, \Delta t_r)}} \min_{\mathbf{X}_{r+1}(t) \in \overline{A(\mathbf{X}_r, \Delta t_r)}} \sum_j \min\{\rho_j^r, \\
\min_{t \in [0, \Delta t_r]} \min_i \|\mathbf{x}_i^{r+1}(t) - \mathbf{y}_j^{r+1}(t)\|\}.
\end{gathered} \tag{2.1}$$

Theorem 1. 1. Formula (2.1) defines the value of the game $\Gamma(\mathbf{X}_0, \mathbf{Y}_0, \sigma_0)$, which is equal to $\underline{V}(\mathbf{X}_0, \mathbf{Y}_0, R_0, \sigma_0)$.

2. The functions $\underline{V}(\mathbf{X}_l, \mathbf{Y}_l, R_l, \sigma_l)$ satisfy Lipschitz conditions in $W_l \times R_+^m$, i.e. for any $(\mathbf{X}_l^1, \mathbf{Y}_l^1, R_l^1)$, $\mathbf{X}_l^2, \mathbf{Y}_l^2, R_l^2) \in W_l \times R_+^m$

$$|\underline{V}(\mathbf{X}_l^1,\mathbf{Y}_l^1,\mathbf{R}_l^1,\sigma_l)-\underline{V}(\mathbf{X}_l^2,\mathbf{Y}_l^2,\mathbf{R}_l^2,\sigma_l)|\leqslant\|\mathbf{R}_l^1-\mathbf{R}_l^2\|+e^{L(T-t_l)}m(\|\mathbf{X}_l^1-\mathbf{X}_l^2\|+\|\mathbf{Y}_l^1-\mathbf{Y}_l^2\|) \quad (2.2)$$

where L is the number occurring in inequalities (1.2).

Proof. The first part of the theorem is proved on the basis of the definition of an ε-equilibrium situation and the definition of $\underline{V}(\mathbf{X}_0, \mathbf{Y}_0, \mathbf{R}_0, \sigma_0)$. Inequality (2.2) will be proved by induction.

Let us prove (2.2) for $l=r$. It follows from the definition of $\underline{V}(\mathbf{X}_r, \mathbf{Y}_r, \mathbf{R}_r, \sigma_r)$ that for any $\varepsilon>0$ controls $\upsilon_j^1(t)$, $u_i^2(t) t\in[0, \Delta t_r]$ exist such that

$$\begin{aligned}
&|\underline{V}(\mathbf{X}_r^1,\mathbf{Y}_r^1,\mathbf{R}_r^1,\sigma_r)-\underline{V}(\mathbf{X}_r^2,\mathbf{Y}_r^2,\mathbf{R}_r^2,\sigma_r)|\leqslant\sum_j(l_{1j}+l_{2j})+\varepsilon\\
&l_{1j}=|\rho_j^{1r}-\rho_j^{2r}| \qquad (2.3)\\
&l_{2j}=\max_{t\in[0,\Delta t_r]}\max_i\Big|\|\mathbf{x}_i^{r+1}(t,\mathbf{x}_i^1,\mathbf{u}_i^2(t))-\mathbf{y}_j^{r+1}(t,\mathbf{y}_j^1,\boldsymbol{\upsilon}_j^1(t))\|-\\
&-\|\mathbf{x}_i^{r+1}(t,\mathbf{x}_i^2,\mathbf{u}_i^2(t))-\mathbf{y}_i^{r+1}(t,\mathbf{y}_i^2,\boldsymbol{\upsilon}_j^2(t))\|\Big|
\end{aligned}$$

Using (1.2), we obtain

$$\begin{aligned}
l_{2j}&\leqslant\max_{t\in[0,\Delta t_r]}\max_i e^{L\Delta t_r}(\|\mathbf{x}_i^1-\mathbf{x}_i^2\|+\|\mathbf{y}_j^1-\mathbf{y}_j^2\|)\leqslant\\
&\leqslant e^{L\Delta t_r}(\|\mathbf{X}_r^1-\mathbf{X}_r^2\|+\|\mathbf{y}_j^1-\mathbf{y}_j^2\|) \qquad (2.4)
\end{aligned}$$

It follows from (2.3) and (2.4) that

$$\begin{aligned}
&|\underline{V}(\mathbf{X}_r^1,\mathbf{Y}_r^1,\mathbf{R}_r^1,\sigma_r)-\underline{V}(\mathbf{X}_r^2,\mathbf{Y}_r^2,\mathbf{R}_r^2,\sigma_r)|\leqslant\\
&\leqslant\|\mathbf{R}_r^1-\mathbf{R}_r^2\|+e^{L\Delta t_r}m(\|\mathbf{X}_r^1-\mathbf{X}_r^2\|+\|\mathbf{Y}_r^1-\mathbf{Y}_r^2\|)+\varepsilon
\end{aligned}$$

Since ε is arbitrary, this gives (2.2) for $l=r$.

Theorem 2. Let σ, $\sigma'\in\Sigma$ and

$$\sigma=\{0,t_1,\ldots,t_l,t_{l+1},\ldots,T\},\ \sigma'=\{0,t_1,\ldots,t_l,t^*,t_{l+1},\ldots,T\}$$

Then

$$\underline{V}(\mathbf{X}_0,\mathbf{Y}_0,\mathbf{R}_0,\sigma)\leqslant\underline{V}(\mathbf{X}_0,\mathbf{Y}_0,\mathbf{R}_0,\sigma') \qquad (2.5)$$

$$|\underline{V}(\mathbf{X}_0,\mathbf{Y}_0,\mathbf{R}_0,\sigma)-\underline{V}(\mathbf{X}_0,\mathbf{Y}_0,\mathbf{R}_0,\sigma')|\leqslant\mu M|t_{l+1}-t_l|,\ \mu=4e^{LT}(m^2+mn) \qquad (2.6)$$

where L and M are the numbers occurring in (1.2).

Proof. Inequality (2.5) is true because team E, playing in the game $\underline{\Gamma}(\mathbf{X}_0, \mathbf{Y}_0, \mathbf{R}_0, \sigma)$, may always guarantee itself the value $\underline{V}(\mathbf{X}_0, \mathbf{Y}_0, \mathbf{R}_0, \sigma')$. Let us prove (2.6). For any $\varepsilon>0$, controls $\mathbf{u}_i^1(t)$, $\mathbf{u}_i^2(t)$, $\boldsymbol{\upsilon}_j^1(t)$, $\boldsymbol{\upsilon}_j^2(t)$ exist such that for $\mathbf{X}_{l+1}^s=(\mathbf{x}_i(\Delta t_l,\ \mathbf{x}_i^l,\ \mathbf{u}_i^s(t)))$, $\mathbf{Y}_{l+1}^s=(\mathbf{y}_j(\Delta t_l,\ \mathbf{y}_j^l,\ \boldsymbol{\upsilon}_j^s(t))$ $(s=1,2)$

$$\begin{aligned}
&0\leqslant\underline{V}(\mathbf{X}_l,\mathbf{Y}_l,\mathbf{R}_l,\sigma_l')-\underline{V}(\mathbf{X}_l,\mathbf{Y}_l,\mathbf{R}_l,\sigma_l)\leqslant\\
&\leqslant\underline{V}(\mathbf{X}_{l+1}^1,\mathbf{Y}_{l+1}^1,\mathbf{R}_{l+1}^1,\sigma_{l+1})-\underline{V}(\mathbf{X}_{l+1}^2,\mathbf{Y}_{l+1}^2,\mathbf{R}_{l+1}^2,\sigma_{l+1})+\varepsilon
\end{aligned}$$

Hence, using (2.2), we obtain

$$0 \leq \underline{V}(\mathbf{X}_l, \mathbf{Y}_l, \mathbf{R}_l, \sigma_l') - \underline{V}(\mathbf{X}_l, \mathbf{Y}_l, \mathbf{R}_l, \sigma_l) \leq$$
$$\leq \left\|\mathbf{R}_{l+1}^1 - \mathbf{R}_{l+1}^2\right\| + e^{LT} m\left(\left\|\mathbf{X}_{l+1}^1 - \mathbf{X}_{l+1}^2\right\| + \left\|\mathbf{Y}_{l+1}^1 - \mathbf{Y}_{l+1}^2\right\|\right) + \varepsilon \tag{2.7}$$

Using (1.2), we obtain

$$\left\|\mathbf{X}_{l+1}^1 - \mathbf{X}_{l+1}^2\right\| \leq 2Mn\left|t_{l+1} - t_l\right|, \ \left\|\mathbf{Y}_{l+1}^1 - \mathbf{Y}_{l+1}^2\right\| \leq 2Mm\left|t_{l+1} - t_l\right|$$
$$\left\|\mathbf{R}_{l+1}^1 - \mathbf{R}_{l+1}^2\right\| \leq \mu M\left|t_{l+1} - t_l\right| / 2$$

Hence

$$0 \leq \underline{V}\left(\mathbf{X}_l, \mathbf{Y}_l, \mathbf{R}_l, \sigma_l'\right) - \underline{V}(\mathbf{X}_l, \mathbf{Y}_l, \mathbf{R}_l, \sigma_l) \leq \mu M\left|t_{l+1} - t_l\right| + \varepsilon$$

This inequality immediately implies the conclusion of the theorem.

Let us consider the quantity $\underline{V}(\mathbf{X}_0, \mathbf{Y}_0) = \sup_{\sigma\in\Sigma} \underline{V}(\mathbf{X}_0, \mathbf{Y}_0, \mathbf{R}_0, \sigma)$ and the partitions $\sigma^{(r)} \in \Sigma$ given by

$$\sigma^{(r)} = \left\{0, \frac{T}{2^r}, \ldots, \frac{2^r - 1}{2^r} T, T\right\} \tag{2.8}$$

Theorem 3.

$$\lim_{r\to\infty} \underline{V}\left(\mathbf{X}_0, \mathbf{Y}_0, \mathbf{R}_0, \sigma^{(r)}\right) = \underline{V}(\mathbf{X}_0, \mathbf{Y}_0)$$

Proof. By Theorem 2, the sequence $\underline{V}(\mathbf{X}_0, \mathbf{Y}_0, \mathbf{R}_0, \sigma^{(r)})$ is non-decreasing, and so has a limit $\lim_{r\to\infty} \underline{V}(\mathbf{X}_0, \mathbf{Y}_0, \mathbf{R}_0, \sigma^{(r)}) = V_0$. Suppose the theorem is false; then $V_0 < \underline{V}(\mathbf{X}_0, \mathbf{Y}_0)$. Choose a partition $\sigma = \{0, t_1, \ldots, t_N, T\}$ so that

$$\underline{V}(\mathbf{X}_0, \mathbf{Y}_0) - \underline{V}(\mathbf{X}_0, \mathbf{Y}_0, \mathbf{R}_0, \sigma) < (\underline{V}(\mathbf{X}_0, \mathbf{Y}_0) - V_0)|3 = \delta|3$$

and a number M_0 such that, for all $r > M_0$

$$\mu MTN2^{-r} < \delta/3$$

Then

$$\underline{V}(\mathbf{X}_0, \mathbf{Y}_0, \mathbf{R}_0, \sigma) \leq \underline{V}\left(\mathbf{X}_0, \mathbf{Y}_0, \mathbf{R}_0, \sigma + \sigma^{(r)}\right)$$
$$\left|\underline{V}\left(\mathbf{X}_0, \mathbf{Y}_0, \mathbf{R}_0, \sigma + \sigma^{(r)}\right) - \underline{V}\left(\mathbf{X}_0, \mathbf{Y}_0, \mathbf{R}_0, \sigma^{(r)}\right)\right| < \delta|3 \tag{2.9}$$

Indeed, the first inequality of (2.9) follows from Theorem 2, and the left-hand side of the second does not exceed

$$\sum_{s=0}^{N-1}\left|\underline{V}\left(\mathbf{X}_0, \mathbf{Y}_0, \mathbf{R}_0, \sigma^{(s+1)}\right) - \underline{V}\left(\mathbf{X}_0, \mathbf{Y}_0, \mathbf{R}_0, \sigma^{(s)}\right)\right|$$

where

$$\sigma^{(0)} = \sigma^{(r)}, \ \sigma^{(1)} = \sigma^{(0)} \cup \{t_1\}, \ldots, \sigma^{(s+1)} = \sigma^{(s)} \cup \{t_s\}, \ \sigma^{(N)} = \sigma + \sigma^{(r)}$$

Using (2.6), we obtain the second inequality of (2.9).

Consequently

$$\underline{V}(\mathbf{X}_0, \mathbf{Y}_0, \mathbf{R}_0, \sigma^{(r)}) \geq \underline{V}(\mathbf{X}_0, \mathbf{Y}_0, \mathbf{R}_0, \sigma) - \delta|3 > V_0 + \delta|3$$

On the other hand, $\underline{V}(\mathbf{X}_0, \mathbf{Y}_0, \mathbf{R}_0, \sigma^{(r)}) \leq V_0$. This contradiction proves the theorem.

We now consider a game $\overline{\Gamma}(\mathbf{X}_0, \mathbf{Y}_0, \sigma_0)$ that differs from $\Gamma(\mathbf{X}_0, \mathbf{Y}_0)$ only in the information state and class of admissible strategies. Let $\sigma_0 \in \Sigma$ (1.3). In the game $\overline{\Gamma}(\mathbf{X}_0, \mathbf{Y}_0, \sigma_0)$, players P_i use POLS as in Definition 1.

Definition 3. A piecewise open-loop co-strategy S_j for player E_j in the game $\overline{\Gamma}(\mathbf{X}_0, \mathbf{Y}_0, \sigma_0)$ is a family of mappings $c_j^l(l=0, 1, \ldots, r)$ which, given the quantities (1.4) and controls $\mathbf{u}_i(t)$, $t \in [t_l, t_{l+1})$, produce a measurable function $\mathbf{v} = \mathbf{v}_j(t) \in V_j$ defined for $t \in [t_1, t_{l+1})$.

Define

$$\overline{V}(\mathbf{X}_l, \mathbf{Y}_l, \mathbf{R}_l, \sigma_l) = \min_{\mathbf{X}_{l+1}(t)\in A(\mathbf{X}_l, \Delta t_l)} \max_{\mathbf{Y}_{l+1}(t)\in A(\mathbf{Y}_l, \Delta t_l)} \overline{V}(\mathbf{X}_{l+1}(\Delta t_l), \mathbf{Y}_{l+1}(\Delta t_l), \mathbf{R}_{l+1}, \sigma_{l+1})$$

$$(\mathbf{X}_l, \mathbf{Y}_l) \in W_l,\ \Delta t_l = t_{l+1} - t_l, \mathbf{R}_{l+1} = (\rho_1^{l+1}, \ldots, \rho_m^{l+1}) \tag{2.10}$$

$$\rho_j^{l+1} = \min\left\{\rho_j^l, \min_{t\in[0,\Delta t_l]} \min_i \left\|\mathbf{x}_i^{l+1}(t) - \mathbf{y}_j^{l+1}(t)\right\|\right\},\ l = 0, 1, \ldots, r-1$$

$$\overline{V}(\mathbf{X}_r, \mathbf{Y}_r, \mathbf{R}_r, \sigma_r) = \min_{\mathbf{X}_{r+1}(t)\in A(\mathbf{X}_r, \Delta t_r)} \max_{\mathbf{Y}_{r+1}(t)\in A(\mathbf{Y}_r, \Delta t_r)} \times$$

$$\times \sum_i \min\left\{\rho_j^r, \min_{t\in[0,\Delta t_r]} \min_i \left\|\mathbf{x}_i^{r+1}(t) - \mathbf{y}_j^{r+1}(t)\right\|\right\}$$

Theorem 1'. 1. Formula (2.10) defines the value of the game $\overline{\Gamma}(\mathbf{X}_0, \mathbf{Y}_0, \sigma_0)$, which is equal to $\overline{V}(\mathbf{X}_0, \mathbf{Y}_0, \mathbf{R}_0, \sigma_0)$.

2. The functions $\overline{V}(\mathbf{X}_l, \mathbf{Y}_l, \mathbf{R}_l, \sigma_l)$ satisfy a Lipschitz condition in $W_l \times R_+^n$, i.e. for any $(\mathbf{X}_l^1, \mathbf{Y}_l^1, \mathbf{R}_l^1)$, $(\mathbf{X}_l^2, \mathbf{Y}_l^2, \mathbf{R}_l^2) \in W_l \times R_+^n$

$$\left|\overline{V}(\mathbf{X}_l^1, \mathbf{Y}_l^1, \mathbf{R}_l^1, \sigma_l) - \overline{V}(\mathbf{X}_l^2, \mathbf{Y}_l^2, \mathbf{R}_l^2, \sigma_l)\right| \leq$$
$$\leq \left\|\mathbf{R}_l^1 - \mathbf{R}_l^2\right\| + e^{L(T-t_l)} m\left(\left\|\mathbf{X}_l^1 - \mathbf{X}_l^2\right\| + \left\|\mathbf{Y}_l^1 - \mathbf{Y}_l^2\right\|\right)$$

where L is the constant occurring in inequalities (1.2).

Theorem 2'. Define $\sigma, \sigma' \in \Sigma$ as in Theorem 2. Then

$$\overline{V}(\mathbf{X}_0, \mathbf{Y}_0, \mathbf{R}_0, \sigma) \geq \overline{V}(\mathbf{X}_0, \mathbf{Y}_0, \mathbf{R}_0, \sigma')$$
$$\left|\overline{V}(\mathbf{X}_0, \mathbf{Y}_0, \mathbf{R}_0, \sigma) - \overline{V}(\mathbf{X}_0, \mathbf{Y}_0, \mathbf{R}_0, \sigma')\right| \leq \mu M \left|t_{l+1} - t_l\right|$$

Theorem 3'.

$$\lim_{r\to\infty} \overline{V}(\mathbf{X}_0, \mathbf{Y}_0, \mathbf{R}_0, \sigma^{(r)}) = \overline{V}(\mathbf{X}_0, \mathbf{Y}_0) = \inf_{\sigma\in\Sigma} \overline{V}(\mathbf{X}_0, \mathbf{Y}_0, \mathbf{R}_0, \sigma)$$

where $\sigma^{(r)} \in \Sigma$ are defined by (2.8).

3. THE MAIN THEOREM AND APPLICATION

Lemma. Let $\sigma', \sigma'' \in \Sigma$. Then

$$\underline{V}(\mathbf{X}_0, \mathbf{Y}_0, \mathbf{R}_0, \sigma') \leq \overline{V}(\mathbf{X}_0, \mathbf{Y}_0, \mathbf{R}_0, \sigma'')$$

The proof is analogous to that of the parallel lemma in [1].

Corollary. $\underline{V}(\mathbf{X}_0, \mathbf{Y}_0) \leq \overline{V}(\mathbf{X}_0, \mathbf{Y}_0)$.

Theorem 4. For any $\varepsilon > 0$, in the game $\Gamma(\mathbf{X}_0, \mathbf{Y}_0)$ an ε-equilibrium situation exists in the class of POLS. The value of the game $\Gamma(\mathbf{X}_0, \mathbf{Y}_0)$ is $\underline{V}(\mathbf{X}_0, \mathbf{Y}_0) = \overline{V}(\mathbf{X}_0, \mathbf{Y}_0)$.

The proof is analogous to that of [1].

Under our assumptions, we can consider a game $\Gamma(\mathbf{X}_0, \mathbf{Y}_0)$ of any duration T. We will denote any such game by $\Gamma(\mathbf{X}_0, \mathbf{Y}_0, T)$, and its value by $V(\mathbf{X}_0, \mathbf{Y}_0, T)$.

Theorem 5. $V(\mathbf{X}_0, \mathbf{Y}_0, t)$,as a function of t is non-increasing in $[0, +\infty)$, and in any interval $[0, T]$ satisfies a Lipschitz condition

$$\left|V(\mathbf{X}_0, \mathbf{Y}_0, t_1) - V(\mathbf{X}_0, \mathbf{Y}_0, t_2)\right| \leq L(T)\left|t_1 - t_2\right|$$

The proof is analogous to that of the parallel theorem in [1].

We will now consider the following game of kind $\gamma(\mathbf{X}_0, \mathbf{Y}_0)$ the goal of team P is to capture all the evaders E_j, while that of team E is to enable at least one of the latter to evade capture.

Definition 4. Considering the game $\gamma(\mathbf{X}_0, \mathbf{Y}_0)$, we shall say that encounter has been avoided if, for any $T > 0$, there exist $\varepsilon(T) > 0$ and POLSs S_j for players E_j, defined in $[0, T]$, such that for any trajectories $x_i(t)$ of players P_i

$$\sum_j \min_{t \in [0,T]} \min_i \left\|\mathbf{x}_i(t) - \mathbf{y}_j(t)\right\| \geq \varepsilon(T) \tag{3.1}$$

Definition 5. We shall say that capture has occurred in the game $\gamma(\mathbf{X}_0, \mathbf{Y}_0)$ if $T > 0$ exist and, for any $\varepsilon > 0$, POLSs Q_i for players P_i, defined in $[0, T]$, such that for any trajectories $y_j(t)$ of players E_j the inequality obtained by inverting the sign of (3.1) holds.

Theorem 6. If a period of time $T > 0$ exists such that $V(\mathbf{X}_0, \mathbf{Y}_0, T) = 0$, then capture occurs in the game $\gamma(\mathbf{X}_0, \mathbf{Y}_0)$; but if $V(\mathbf{X}_0, \mathbf{Y}_0, T) > 0$ for all $T > 0$, then encounter can be avoided in the game $\gamma(\mathbf{X}_0, \mathbf{Y}_0)$.

Proof. Let $V(\mathbf{X}_0, \mathbf{Y}_0, T) = 0$. Then, operating as in the game $\Gamma(\mathbf{X}_0, \mathbf{Y}_0, T)$, team P can guarantee itself the value $V(\mathbf{X}_0, \mathbf{Y}_0, T)$ to within any degree of accuracy. Hence capture must occur in the interval $[0, T]$.

The proof of the second part of the theorem is similar.

Remark. The possibility of evasion in the game $\gamma(\mathbf{X}_0, \mathbf{Y}_0)$in the sense of Definition 5 does not imply the possibility of evasion over the interval $[0, +\infty)$.

Example. Consider the following game in R^2 between pursuers P_i $(i = 1, 2, 3, 4)$ and an evader E. The laws of motion are

$$\dot{\mathbf{x}}_i = \mathbf{u}_i, \ \|\mathbf{u}_i\| \leq 1, \ i = 1,2,3, \ \|\mathbf{u}_4\| \leq \beta < 1$$

$$\dot{\mathbf{y}} = \mathbf{v}, \ \|\mathbf{v}\| \leq 1$$

the initial positions are

$$\mathbf{x}_1^0 = (-1,0), \ \mathbf{x}_2^0 = (1,0), \ \mathbf{x}_3^0 = (0,-1), \ \mathbf{x}_4^0 = (0,3), \ \mathbf{y}^0 = (0,0)$$

Then encounter can be avoided in the game $\gamma(\mathbf{X}_0, \mathbf{Y}_0)$ in the sense of Definition 5, but the game $\gamma(\mathbf{X}_0, \mathbf{Y}_0)$ cannot be terminated in E's favour in the interval $[0, +\infty)$ (see [8]†).

Let $\Gamma_0(\mathbf{X}_0, \mathbf{Y}_0)$ denote the following game of degree. In each situation (Q_i, S_j), where Q_i and S_j are POLS for players P_i and E_j, define the value of the payoff function $T(Q_1, \ldots, S_m)$ to be the first time at which all the evaders E_j have been successfully captured. If capture does not occur in the situation (Q_i, S_j), define $T(Q_1, \ldots, S_m)=\infty$. Team P endeavours to minimize $T(Q_1, \ldots, S_m)$ and team E to maximize it. We have the following.

Theorem 7. Suppose that T_0, the first time at which $V(\mathbf{X}_0, \mathbf{Y}_0, T_0)=0$, exists. Then the value of the game $\Gamma(\mathbf{X}_0, \mathbf{Y}_0)$ exists and is T_0.

The results of this paper may be extended to other classes of many-person differential games.

REFERENCES

1. PETROV N. N., The existence of the value of a game of pursuit. *Differents. Uravn.* **7**, 5, 827–839, 1971.
2. KRASOVSKII N. N. and SUBBOTIN A. I., *Positional Differential Games.* Nauka, Moscow, 1974.
3. SUBBOTIN A. I. and CHENTSOV A. G., *Optimization of Guarantee in Control Problems.* Nauka, Moscow, 1981.
4. KRASOVSKII N. N., *Control of a Dynamical System*. Nauka, Moscow, 1985.
5. PETROSYAN L. A., *Differential Games of Pursuit*. Izd. Leningrad. Gos. Univ., 1977.
6. TOMSKII G. V. and ULANOV V. A., *Games in General Control Systems.* Izd. Irkutsk. Univ., Irkutsk, 1987.
7. GABRIELYAN M. A. and SUBBOTIN A. I., Game problems of encounter with m target sets. *Prikl. Mat. Mekh.* **43**, 2, 204–208, 1979.
8. CHERNOUS'KO F. L., A problem of escape from several pursuers. *Prikl. Mat. Mekh.* **40**, 1, 14–25, 1976.

Translated by D.L.

†See also PETROV N. N., Some problems of evasion in differential games. Candidate dissertation, 12.06.86. Leningrad State University, Leningrad, 1986.